Exploring the Meaning of Rational Exponents

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Target Grade: Algebra 2

Time Required: 60 minutes

Standards

Common Core Math Standard

- **HSN-RN.A.1** Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.

Lesson Objectives

Students will:

- Be able to model exponential growth, speculate that $x^{1/n} = \sqrt[n]{x}$, and explain this equality using the idea of uniqueness ($(x^{1/n})^n = x = \sqrt[n]{x}^n$).

Central Focus

Exponential growth is all around students in the growth of diseases, population, and interest. This lesson incorporates the exponential growth of wildfire, a phenomenon that students see on the news. Students will build on their prior knowledge of the law of exponents to model exponential growth.

Key Terms: exponential growth, rational exponents, exponents

Background Information

The only text used in this lesson is in the group activity attachment written for this lesson. This activity is a modified version of the activity found at http://tasks.illustrativemathematics.org/content-standards/tasks/385 . The language has been simplified so that many different students will be able to access the problems.

In this lesson, teachers may show their students the video about the wildfire: https://www.redding.com/videos/news/local/fires/2020/01/28/nasa-satellite-carr-fire-time-lapse-how-2018-california-wildfire-spread/4594015002/ However, teachers may use a video of a more current wildfire.

Students will participate in a number talk in this lesson. Number talk is an activity in which students have the opportunity to use their mental math skills. Students develop efficiency in calculating expressions that may seem hard to solve without pencil and paper. Here is a link to a video with more information about number talks: https://www.youtube.com/watch?v=oaAcZ2zBijU

In this lesson, students will need to find the exact answer to the questions. The exact answer is not rounded. The image below shows an example of an exact answer.

Exact Answers: $\boxed{x = \frac{-5 \pm \sqrt{13}}{6}}$

Rounded Answers: $\boxed{x = -.23, x = -1.43}$

Figure 1: https://www.katesmathlessons.com/quadratic-formula.html

Prior to this lesson, students will need to be familiar with the laws of exponents to quickly calculate exponents such as 2^6.

Exponent Rules For $a \neq 0, b \neq 0$	
Product Rule	$a^x \times a^y = a^{x+y}$
Quotient Rule	$a^x \div a^y = a^{x-y}$
Power Rule	$\left(a^x\right)^y = a^{xy}$
Power of a Product Rule	$(ab)^x = a^x b^x$
Power of a Fraction Rule	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
Zero Exponent	$a^0 = 1$
Negative Exponent	$a^{-x} = \frac{1}{a^x}$
Fractional Exponent	$a^{\frac{x}{y}} = \sqrt[y]{a^x}$

Figure 2: https://www.onlinemathlearning.com/exponents-scientific-notation.html

Prior to this lesson, teachers will need to be familiar with the laws of exponents. Teachers will also need to be familiar with exponential growth and be able to model it.

- Laws of Exponents
  - Here is a video that talks about Laws of Exponents: https://youtu.be/LkhPRz7Hocg
    - $x^1 = x$
    - $x^0 = 1$
    - $x^{-n} = \frac{1}{x^n}$
    - $(x^m)^n = x^{mn}$
    - $x^m x^n = x^{m+n}$
    - $\frac{x^m}{x^n} = x^{m-n}$
    - $(xy)^m = x^m y^m$
    - $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

- Exponential growth
  - Exponential growth is the idea that something always grows in relation to its current value, such as always doubling.
    (Exponential Growth and Decay (mathsisfun.com))

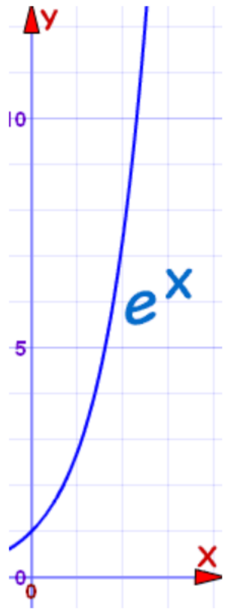


Figure 3: https://www.mathsisfun.com/algebra/exponential-growth.html

Materials

- Only You Can Calculate the Damage of a Wildfire
- Rational Exponents Homework
- Teacher's notes (for the lesson)
- Paper
- Pencil
- Calculator (students may use Desmos: https://www.desmos.com/)
- Video link: https://www.redding.com/videos/news/local/fires/2020/01/28/nasa-satellite-carr-fire-time-lapse-how-2018-california-wildfire-spread/4594015002/

Instruction

Before class begins, write the expression 2^6 on the board.

Introduction (5-10 minutes)

- As a warmup, have the students participate in a number talk to solve the expression 2^6 .
    - Number talk is an activity in which students have the opportunity to use their mental math skills. Students develop efficiency in calculating expressions that may seem hard to solve without pencil and paper.
    - Students will use prior knowledge of integer exponents to evaluate the expression 2^6 without pen and paper and without multiplying 2 six times.
- After the students have time to think individually, the class will discuss different methods they used to calculate this quantity.
    - Students may misremember the rules and think that $a^n a^m = a^{nm}$.
- After discussing the different methods, have a whole class discussion of the laws of exponents and why they are true.

Activity (45 minutes)

- Hand out the wildfire activity worksheet.

- Put the students in groups of 3-4. Instruct the students to work through the wildfire activity in which they will show that $x^{1/n} = \sqrt[n]{x}$. Some students may be able to calculate the values by hand. However, for all parts of this activity, students will be told that they can use their graphing calculators or Desmos if they wish.
- Have students work on problem 1.

- As groups work on problem 1, the teacher will monitor their solutions.
  - 1. (a) Find how many acres the fire has covered after 20 hours without extending the table.
  - 1. (b) Write a description of how to find the number of acres after any amount of time. Justify your reasoning.
- After problem 1 in the activity, as a class, discuss each group's representations of the problem.
  - Any misconceptions (the misconceptions are also outlined in the attachments) will be discussed first.
    - When finding the new multiplier for half-hour intervals, students may use x^2 instead of x*x. They must be directed to see that for both half-hour intervals, the multiplier is multiplied twice, not added. A similar situation could occur for the 20-minute interval.
    - Students might not realize that when asked for an exact answer, they cannot answer with rounded decimal places. The teacher will also reinforce this when going around the room during the activity.
  - Next, if a group makes a graph, they will present their work.
  - Finally, the last student work to be discussed will be equations or verbal descriptions of the model.
  - Use this discussion to help the class come to a consensus on the appropriate equation to use for the next part of the activity.
- Have students work on problems 2 and 3.
  - 2. (a) Every hour the size increases by a factor of 2. By what factor does the size of the fire increase every half hour? Give an exact answer. (Hint: Going forward a half-hour twice has the same effect of going forward 1 hour)
  - 2. (b) Find the number of acres on fire after the first 30 minutes using this value. What does this tell you about the value of $2^{1/2}$? (Hint: Find the number of acres using the equation.)
  - 2. (c) What would the factor be for every 20-minute interval? Give an exact answer.
  - 2. (d) Find the number of acres on fire after the first 20 minutes using this value. What does this tell you about the value of $2^{1/3}$?
  - 3. Hypothesize and justify a general rule for the relationship between rational exponents and radicals.
- Monitor their solutions.
  - Students might not understand that they can raise a number to a rational exponent. For part 2b, be available to answer questions and clear up confusion.

- After groups have time to work on problems 2 and 3, the class will come together for another discussion.
    - This discussion will explore the conclusion of the problem 2 that $2^{1/2} = \sqrt{2}$ and $2^{1/3} = \sqrt[3]{2}$.
    - The first representation of this problem that will be selected for discussion is a table (if any groups make one).
        - This work gives an approximation but does not give an exact value.
    - The next student work to be selected for discussion would be any diagram drawn by the students.
        - From the diagram, the class will discuss how to represent the problem in an equation format, which will give an exact answer.
    - As a class, discuss the misconception of adding the factors instead of multiplying them.
- If any groups finish the activity sheet early, the teacher can show them the video found at https://www.redding.com/videos/news/local/fires/2020/01/28/nasa-satellite-carr-fire-time-lapse-how-2018-california-wildfire-spread/4594015002/.
    - With this video, the teacher can ask the group if they think the model in our activity corresponds to real life wildfires.
    - The teacher can also ask what factors would need to be taken into account to make a more accurate model.
    - This activity requires high cognitive thinking as students must evaluate models and identify important real-life factors that affect the system.

Figure 4: https://www.upi.com/Top_News/US/2022/07/10/California-washburn-fire-giant-sequoias-yosemite-park/8511657481069/

- Throughout the activity here are some potential questions the teacher may ask:
  - What are different ways you could represent the growth of the wildfire? (graph, table, equation, picture)
  - Which representation helps find an exact value at any point in time?
    (Students should cite evidence that an equation helps find an exact value because a table only gives you a limited number of points, and the values of graphs can be approximated but not found exactly.)
  - How many ways can you write an equation for this situation? Which equation is easiest to use?
    (Students must use the laws of exponents to rewrite the equation in different forms (ex. $2^{(t+2)} = 2^2*2^t = 4*2^t$))
  - How do you find an unknown value when you need it to be exact? (For problem 2a)
    (Students should cite that graphs are good for finding approximations of unknown values, but that equations are best for finding exact values.)
  - What is a factor? Do you multiply by the factor or add it?
  - How do you get from the starting size (4 acres) to the size at 30 minutes? Then, how do you take this value and get to the size at 1 hour? (Same questions can be repeated for the 20-minute intervals) Are you multiplying by the factor twice or adding it twice?
    (Students should cite the fact that we multiply by the factor twice (once for the first 30 minutes and again for the second 30 minutes) to show that we need $x*x = x^2$ instead of $x+x= 2x$.)
  - How would you represent 30 minutes in terms of an hour?
    (Students should see that here t = ½. They will be able to plug this into the equation and use a calculator to find the value for 2b)
  - What is the relationship between $2^{1/2}$ and $\sqrt{2}$?
    (Students must cite that the exponent corresponds to orders of roots. Students should use their answer for 2a and the equation they derived for 1 to show that since $4 \cdot 2^{1/2} = 4 \cdot \sqrt{2}$, $2^{1/2} = \sqrt{2}$.)
  - What is another way to show that $2^{1/2} = \sqrt{2}$?
    (Students should see that both quantities satisfy the equation $x^2=2$, so they must be the same.)
  - For question 3: What patterns do you see? Is there any relationship between rational exponents and radicals? Can you extend the pattern to other numbers? How are rational exponents and radicals related to the equation $y^n = x$?

Closure (5-10 minutes)

- Have a whole class discussion focusing on problem 3 in which students are asked to make a general rule for rational exponents and explain why it works.
    - In this discussion, students will show if they understand that rational exponents correspond to radicals, and they must justify their thinking.
        - Students might be able to understand this rule, but they might not know how to show or explain it.
        - The final discussion should guide students to the understanding that $x^{1/n} = \sqrt[n]{x}$.
        - A hint given by the teacher could be to look at the equation $y^n = x$.
            - What could y be here?
            - This hint should also help students with their homework.
- As students leave, hand out a short worksheet for homework.
    - The homework will ask questions that evaluate whether the student understands and can explain why $x^{1/n} = \sqrt[n]{x}$.

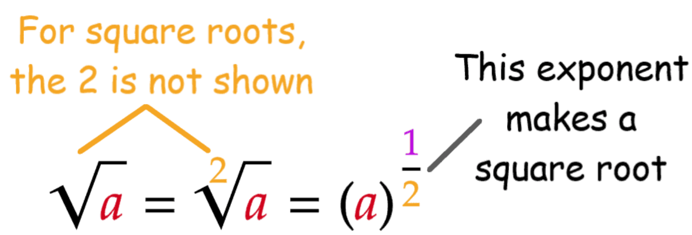


Figure 5: https://www.bing.com/images/search?view=detailV2&ccid=TecW7wyX&id=00D798243A48F6AB99A907296716F9482055275E&thid=OIP.TecW7wyXau0JAIu8b-sJ5gHaC5&mediaurl=https%3a%2f%2fmatterofmath.com%2fwp-content%2fuploads%2f2021%2f01%2f3-Ways-of-Showing-Square-Roots.p

Differentiation

Advanced Students

- The students can complete the extra activity by watching the video linked above.
    - For the extra activity: Based on this video of a real wildfire spread, do you think the model in our activity accurately describes real wildfires? What are practical, or environmental, factors that we would need to take into account to make a more accurate model? (Examples: geography, possible bounds of the fire/carrying capacity, wind direction and speed, oxygen levels)
- If one group is excelling, then they can present the more difficult sections of the task or on the extra activity.

Students who are struggling

- The first question should be familiar to most students, so the group that struggles with the other half of the activity can present their reasoning on the first portion of the task.
- Because the teacher is not directly lecturing during this activity, he/she is free to move about the room to help any students who need accommodations.
- Students do not need to answer all of the questions, so they can focus on one question if they need.

Grouping

- Students will have different roles such as calculating or writing in the group work helps cater to their different interests and skills.
- The groups will be heterogeneous, so each group will have students of different levels.

Assessment

Formative Assessment

- Throughout the lesson, the teacher will ask questions to the students that force them to use their prior knowledge of exponents and exponential functions to reason their way to a conclusion.
- In the discussions, students will show if they understand that rational exponents correspond to radicals, and they must justify their thinking.

Summative Assessment

- The discussions at the end of the group work as well as the homework will show how well the students are comprehending the connection between rational exponents and radicals.
  - If they are understanding this relationship well, then future instruction will be used to expand on calculations and manipulations of rational exponents. For example, students would move on to learn how to use the form $(a^{1/m})^n = \sqrt[m]{a}^n$.
  - If the students are struggling with the concept, then further work will be put in to have students learn how to prove that $x^{1/n} = \sqrt[n]{x}$.
- The results of the group activity will inform how this lesson is taught in the future.
  - In particular, how students work through the group activity will reveal whether the questions need to be rewritten for more simplicity, and whether the hints should be kept or taken out to increase the cognitive demand.

- Evidence for student learning will be collected through the group worksheets as well as the homework.
    - Both activities require students to analyze, interpret, and evaluate information. Their justifications of their thinking require that they draw conclusions, generalize, and produce arguments.
- The homework activates students' reasoning and requires a higher level of cognition as they connect laws of exponents to rational exponents and give the reasoning behind their thinking.

Only You Can Calculate the Damage of a Wildfire

Name: ___________________________

Wildfires can spread like crazy and are hard to put out. On a hot day at a large farm, Sam was trying to see if he could set a piece of grass on fire with a magnifying glass. Unfortunately, his experiment worked. At 9 am, his blade of grass caught on fire and started to spread. By 10 am, 4 acres of the surrounding land are on fire. Every hour, the size of the fire doubles. Taking 10:00 am to be hour zero, approximately how many hours will it take until the fire has covered the entire 500-acre farm? You can use the table provided:

Hour (t)	0	1	2	3	4	5	6	7	8
Acres (A)	4								

1. (a) Find how many acres the fire has covered after 20 hours without extending the table.

(b) Write a description of how to find the number of acres after any amount of time. Justify your reasoning.

2. (a) Every hour the size increases by a factor of 2. By what factor does the size of the fire increase every half hour? Give an exact answer. (Hint: Going forward a half-hour twice has the same effect of going forward 1 hour)

(b) Find the number of acres on fire after the first 30 minutes using this value. What does this tell you about the value of $2^{1/2}$? (Hint: Find the number of acres using the equation.)

(c) What would the factor be for every 20-minute interval? Give an exact answer.

(d) Find the number of acres on fire after the first 20 minutes using this value. What does this tell you about the value of $2^{1/3}$?

3. Hypothesize and justify a general rule for the relationship between rational exponents and radicals.

Rational Exponents Homework

Name: ____________________

Calculate these quantities:

1. $16^{1/2} =$

2. $4^{1/4} \cdot 4^{1/4} =$

3. $27^{1/3} =$

4. Is this statement true or false? Explain why: $3^{1/4} = \sqrt[4]{3}$

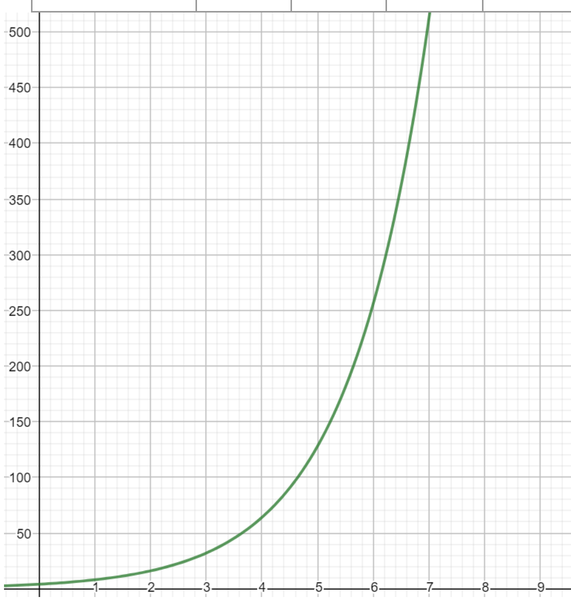
Teacher's notes:

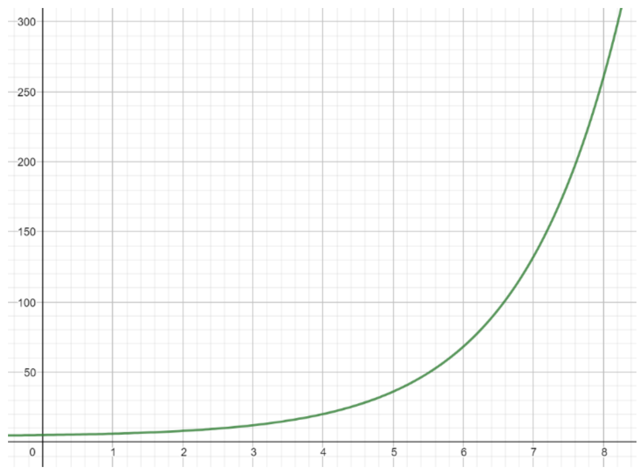
1. After 1, stop for discussion. Different groups will be asked to discuss their findings, starting will graphs and then moving to equations. If there are any misconceptions, these will be addressed first. The class will discuss how they arrived at their answers and how to derive an equation.

The equation can be written in the form $2^{(t+2)} = 2^2*2^t = 4*2^t$.

If students graph the situation, they will see that at t = ½, the number of acres is not halfway between 4 and 8, so the relationship is not linear.

Hour	0	1	2	3	4	5	6	7	8	9
Acres	4	8	16	32	64	128	256	512	1024	2048





Possible solutions:

$A = 4 * 2^t = 2^{t+2}$

Possible Miconception : $A = 4 + 2^t$

Another possible miconception: Students do not understand that the variable should be exponent and not the base. Some forms of this misconception could result in equations like $A = 4 * t^2 \text{ or } A = 4 + t^2$

2a. If x is our multiplier for every half hour, then in every hour, the population will be multiplied by x twice. So, $x^2 = 2$, our multiplier for every hour. Then $x = \sqrt{2}$.

Misconception: We cannot simply add 2 half-hour intervals to make a full hour. It is incorrect here to use the equation 2x = 2 so that x = 1. The population has changed within each half hour, so P_0 (the population at t = 0) is not the same as $P_{0.5}$ (the population at the first half hour). So P_0*x is not equal to $P_{0.5}$*x. You could find the population after 1 hour by doing (P_0*x)*x.

Possible solutions:

Correct: $y \cdot y = 2\ y^2 = 2\ y = \sqrt{2}$

Misconception:

$y + y = 2\ 2y = 2\ y = 1$

Hour	0	1/2	1	3/2	2
Acres	4	5.657	8	11.313	16

Here, each half hour, the acres increase by a factor of about 1.41. (What exact value does it represent?)

2b. Since the factor is $\sqrt{2}$, the population after the first half hour would be $4 \cdot \sqrt{2}$. The table above can also be used to find an approximation of this value. Using the equation, the number of acres after the first half hour would be $4*2^{1/2}$. Since these values are equal ($4 \cdot \sqrt{2} = 4 \cdot 2^{1/2}$), $\sqrt{2} = 2^{1/2}$.

2c. If y is our factor every 20 minutes, then for a full hour, we must multiply by y three times: $y*y*y = y^3$. This quantity must be equivalent to the factor for a full hour so $y^3 = 2$ and $y = \sqrt[3]{2}$.

Solutions:

Correct:

$$y \cdot y \cdot y = 2\ y^3 = 2\ y = \sqrt[3]{2}$$

Misconception:

Hour	0	1/3	2/3	1	4/3
Acres	4	5.04	6.35	8	10.08

$$y + y + y = 2\ 3y = 2\ y = \frac{3}{2}$$

Here, the factor for every 20 min is about 1.26. (What exact value does this represent?)

2d. After the first 20 minutes, the number of acres on fire would be $4 * \sqrt[3]{2}$. Our equation would give us the acres after 20 minutes to be $4*2^{1/3}$. By the same reasoning as above, $2^{1/3} = \sqrt[3]{2}$.

3. From here students should be able to generalize the rule to find that $x^{1/n} = \sqrt[n]{x}$. Some questions to ask to get them to this generalization:

Do you see any patterns in the answers of 2b and 2d?

Does this pattern work for other numbers?

What do you think these answers tell us about the relationship between rational exponents and radicals?